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Nonlocality and entanglement in a strange system

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Abstract. We show that the relation between nonlocality and entanglement is subtler than one naively expects. In order to do this we consider the neutral kaon system – which is oscillating in time (particle–anti-particle mixing) and decaying – and describe it as an open quantum system. We consider a Bell–CHSH inequality and show a novel violation for non-maximally entangled states. Considering the change of purity and entanglement in time we find that, despite the fact that only two degrees of freedom at a certain time can be measured, the neutral kaon system does not behave like a bipartite qubit system.

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1 Introduction

In the last years many experiments have been performed which confirm the peculiar predictions of the quantum theory, in particular the existence of correlations of phenomena that manifest themselves at two different locations and fail to be explainable by local realistic theories. As a powerful tool to detect nonlocality – which ensures secure communication, see e.g. [1] – are the famous Bell inequalities. On the other hand there is currently a huge effort going on to develop entanglement measures using the quantum physics tools of Hilbert space, observables and tensor products. A lot of different measures have been proposed so far, and only for bipartite qubit systems the problem has satisfactorily been solved. Yet, very recently a third approach has been proposed: it asks how huge non-local correlations can be only assuming non-signaling (no faster-than-light communication). Or differently stated: one may wonder why nature is not more non-local than predicted by quantum theory; see e.g. [2, 3].

The purpose of this letter is to shed light on the features of nonlocality and entanglement for massive meson–anti-meson systems, in particular for the neutral kaon–anti-kaon system. The neutral K-mesons (or simply: kaons) are bound states of quarks and anti-quarks, or, more precise, the strangeness state $+1$, K^0, is composed of an anti-strange quark and a down quark and the strangeness state -1, $\bar{K}^0$, is composed of a strange and anti-down quark.

Interestingly, also for strange mesons entangled states can be obtained, in analogy to the entangled spin up and down pairs, or H and V polarized photon pairs. Such states are produced by e^+e^--colliders through the reaction $e^+e^- \to \Phi \to K^0\bar{K}^0$, in particular at DAΦNE in Frascati, or they are produced in $p\bar{p}$-collisions, like, e.g., at LEAR at CERN. There, a $K^0\bar{K}^0$ pair is described at the time $t=0$ by the entangled antisymmetric Bell state

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left\{|K^0\rangle_{\mathrm{l}}\otimes|\bar{K}^0\rangle_{\mathrm{r}} - |\bar{K}^0\rangle_{\mathrm{l}}\otimes|K^0\rangle_{\mathrm{r}}\right\}, \qquad (1)$$

where l denotes the particle moving to the left hand side and r the particle moving to the right hand side.

Analogously to entangled photon systems, for these systems Bell inequalities can be derived, i.e. the most general Bell inequality of the CHSH-type is given by [4]

$$\begin{aligned} S_{k_n,k_m,k_{n'},k_{m'}}(t_1,t_2,t_3,t_4) = \\ \left|E_{k_n,k_m}(t_1,t_2) - E_{k_n,k_{m'}}(t_1,t_3)\right| \\ + \left|E_{k_{n'},k_m}(t_4,t_2) + E_{k_{n'},k_{m'}}(t_4,t_3)\right| \le 2\,. \end{aligned} \qquad (2)$$

Here Alice can choose on the kaon propagating to her left hand side the "quasi-spin", i.e. a certain superposition of kaon and anti-kaon $|k_n\rangle = \alpha|K^0\rangle + \beta|\bar{K}^0\rangle$, and how long the kaon propagates, the time t. The same options are given to Bob for the kaon propagating to the right hand side. As in the usual photon setup, Alice and Bob can choose among two settings. The expectation value $E_{k_n,k_m}(t_1,t_2)$ then concerns Alice choosing to measure the quasi-spin k_n at time t_1 on the kaon propagating to her side as Bob chooses to measure k_m at time t_2 on his kaon.

We notice now already that in the neutral kaon case we have more options than in the photon case; we can vary in the quasi-spin space or vary the detection times or both.

Let us first choose all times equal to zero and choose the quasi-spin states $k_n = K_{\mathrm{S}}, k_m = \bar{K}^0, k_{n'} = k_{m'} = K_1^0$, where K_{S} is the short-lived eigenstate, one eigenstate of the time evolution, and K_1^0 is the $\mathcal{CP}$ plus eigenstate. Here $\mathcal{C}$ stands for charge conjugation and $\mathcal{P}$ for parity. The neutral

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kaon system is known for violating the combined transformation $\mathcal{CP}$. In [5] the authors show that after optimizing the Bell inequality (BI) can be turned into

$$\delta \leq 0\,, \tag{3}$$

where δ is the $\mathcal{CP}$ violating parameter in the mixing. Experimentally, δ corresponds to the leptonic asymmetry of kaon decays which is measured to be $\delta = (3.27 \pm 0.12)10^{-3}$. This value is in clear contradiction to the value required by the BI above, i.e. by the premises of local realistic theories. In this sense the violation of a symmetry in high energy physics is connected to the violation of a Bell inequality, i.e. to nonlocality.[1] This is clearly not the case for photons; they do not violate the $\mathcal{CP}$ symmetry. Moreover, the $\mathcal{CP}$ violating parameter is measured for single states, but it nevertheless gives information on bipartite states.

Although the BI (3) is as loophole free as possible, the probabilities or expectations values involved are not directly measurable, because experimentally there is no way to distinguish the short-lived state K_{S} from the $\mathcal{CP}$ plus state K_1^0 directly.

In this work we want to investigate another choice for the Bell inequality (2), i.e. all quasi-spins are equal to those for $\bar{K}^0$, and we are going to vary all four times:

$$\begin{aligned} &S_{\bar{K}^0,\bar{K}^0,\bar{K}^0,\bar{K}^0}(t_1,t_2,t_3,t_4) \\ &\quad = |E_{\bar{K}^0,\bar{K}^0}(t_1,t_2) - E_{\bar{K}^0,\bar{K}^0}(t_1,t_3)| \\ &\quad + |E_{\bar{K}^0,\bar{K}^0}(t_4,t_2) + E_{\bar{K}^0,\bar{K}^0}(t_4,t_3)| \leq 2\,. \end{aligned} \tag{4}$$

This has the advantage that it can in principle be tested in experiments: Alice and Bob insert at a certain distance from the source (corresponding to the detection times) a piece of matter forcing the incoming neutral kaon to react. Because the strong interaction is strangeness conserving one obtains via the reaction products knowledge as to if the incoming particle is an anti-kaon or not. Note that different to photons a *no* event does not mean that the incoming kaon is a K^0 but also includes the case that it has decayed before. In principle, the strangeness content can also be obtained via decay modes, but Alice and Bob have no way to force their kaon to decay at a certain time; the decay mechanism is a spontaneous event. However, a necessary condition to refute any local realistic theory is that we are to have *active* measurements, i.e. ones exerting the free will of the experimenter (for more details consult [6]).

Our question is: can we violate the Bell–CHSH inequality sensitive to strangeness, (4), for a certain initial state, and what is the maximum value?

The first naive guess would be yes. In [4, 6] the authors studied the problem for the initial maximally entangled Bell state, (1), and they found that a value greater than 2 cannot be reached, i.e. one cannot refute any local realistic theory. The reason is that the particle–anti-particle oscillation is too slow compared to the decay or vice versa, i.e., the ratio of oscillation to decay, $x = \frac{\Delta m}{\Gamma}$, is about 1 for kaons and not 2, as is necessary for a formal violation. A different view is that the decay property acts as a kind of "decoherence", as we will show. From decoherence studies we know that some states are more "robust" against a certain kind of decoherence than others, and this leads to the question if another maximally entangled Bell state or maybe a different initial state would lead to a violation.

In order to proceed let us study first how single neutral kaons are handled via open quantum systems, and then discuss entangled kaons and their entanglement and purity properties, which gives us insight in the behavior of this strange two-state system. The formalism also enables us to calculate the correct expectation values for the arbitrary initial states needed for the Bell inequality (4).

Different kinds of Bell inequalities are discussed e.g. in [7–9] and also decoherence models can be investigated, e.g. [10, 11], and the model proposed in the former reference has recently been tested via experimental data [12, 13].

2 Open quantum formalism of decaying systems

Neutral kaons are a decaying two-state system due to the particle–anti-particle oscillation in time and are usually described via an effective Schrödinger equation which we write in the Liouville–von Neumann form

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\mathrm{ss}} = -\mathrm{i}H_{\mathrm{eff}}\rho_{\mathrm{ss}} + \mathrm{i}\rho_{\mathrm{ss}}H_{\mathrm{eff}}^{\dagger}\,, \tag{5}$$

where ρ_{ss} is a 2×2 matrix and the Hamiltonian H_{eff} is non-Hermitian. Using the Wigner–Weisskopf approximation, the effective Hamilton can be calculated to be $H_{\mathrm{eff}} = H - \frac{\mathrm{i}}{2}\Gamma$, where the mass matrix H and the decay matrix Γ are both Hermitian and positive. Here the weak interaction Hamiltonian responsible for decay is treated as a perturbation, and interactions between the final states are neglected. This Wigner–Weisskopf approximation gives the exponential time evolution of the two diagonal states of H_{eff}:

$$|K_{\mathrm{S/L}}(t)\rangle = \mathrm{e}^{-\mathrm{i}m_{\mathrm{S/L}}t}\mathrm{e}^{-\frac{\Gamma_{\mathrm{S/L}}}{2}t}|K_{\mathrm{S/L}}\rangle\,, \tag{6}$$

where $m_{\mathrm{S/L}}$ and $\Gamma_{\mathrm{S/L}}$ are the masses and decay constants for the short/long-lived state $K_{\mathrm{S/L}}$ ($\Gamma_{\mathrm{S}} \approx 600\Gamma_{\mathrm{L}}$; $\Delta m = m_{\mathrm{L}} - m_{\mathrm{S}} \simeq \Gamma_{\mathrm{S}}/2$). A kaon with strangeness $+1$ (kaon) or -1 (anti-kaon) is a superposition of the two mass-eigenstates, i.e. $|K^0\rangle \simeq \frac{1}{\sqrt{2}}\{|K_{\mathrm{S}}\rangle + |K_{\mathrm{L}}\rangle\}$ and $|\bar{K}^0\rangle \simeq \frac{1}{\sqrt{2}}\{-|K_{\mathrm{S}}\rangle + |K_{\mathrm{L}}\rangle\}$. Here the small $\mathcal{CP}$ violation is safely neglected throughout the letter. What makes the neutral kaon systems so attractive for many physical analyses is the huge factor between the two decay rates, i.e. $\Gamma_{\mathrm{S}} \approx 600\Gamma_{\mathrm{L}}$, and that the strangeness oscillation is $\Delta m = m_{\mathrm{L}} - m_{\mathrm{S}} \simeq \Gamma_{\mathrm{S}}/2$.

Considering (6) we notice that the state is not normalized for $t > 0$. Indeed, we are not describing a system; for $t > 0$ a neutral kaon has a surviving and decaying component. In [14] the authors show that by enlarging the

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[1] That is to say, for all times equal to zero rather contextuality than nonlocality is tested. However, it has been shown [4] that also for $t \geq 0$ the BI is violated.

original two-dimensional Hilbert space by at least two further dimensions representing the decay product states, the non-Hermitian part of $H_{\rm eff}$ can be incorporated into the dissipator of the enlarged space via a Lindblad operator. Thus the time evolution of neutral kaons is described by an open quantum formalism, in particular by a master equation [15, 16]:

$$\frac{\rm d}{{\rm d}t}\rho = -{\rm i}[\mathcal{H},\rho] - \mathcal{D}[\rho]\,, \tag{7}$$

where the dissipator under the assumption of complete positivity and Markovian dynamics has the well known general form $\mathcal{D}[\rho]=\frac{1}{2}\sum_j(\mathcal{A}_j^\dagger\mathcal{A}_j\rho+\rho\mathcal{A}_j^\dagger\mathcal{A}_j-2\mathcal{A}_j\rho\mathcal{A}_j^\dagger)$. The density matrix ρ lives on $\mathbf{H}_{\rm tot}=\mathbf{H}_{\rm s}\bigoplus\mathbf{H}_{\rm f}$, where s and f denote "surviving" and "decaying" or "final" components, and it has the following decomposition:

$$\rho = \begin{pmatrix}\rho_{\rm ss} & \rho_{\rm sf}\\ \rho_{\rm sf}^\dagger & \rho_{\rm ff}\end{pmatrix}\,, \tag{8}$$

where ρ_{ij} with $i,j=$ s,f denote 2×2 matrices. The Hamiltonian $\mathcal{H}$ is the Hamiltonian H of the effective Hamiltonian $H_{\rm eff}$ extended to the total Hilbert space $\mathbf{H}_{\rm tot}$, and Γ of $H_{\rm eff}$ defines a Lindblad operator by $\Gamma=A^\dagger A$, i.e.

$$\mathcal{H}=\begin{pmatrix}H & 0\\ 0 & 0\end{pmatrix},\quad \mathcal{A}=\begin{pmatrix}0 & 0\\ A & 0\end{pmatrix}\quad \text{with}\quad A:\mathbf{H}_{\rm s}\to\mathbf{H}_{\rm f}\,.$$

Rewriting the master equation for ρ, (8), on $\mathbf{H}_{\rm tot}$,

$$\dot\rho_{\rm ss} = -{\rm i}[H,\rho_{\rm ss}] - \frac{1}{2}\{A^\dagger A,\rho_{\rm ss}\}\,, \tag{9}$$

$$\dot\rho_{\rm sf} = -{\rm i}H\rho_{\rm sf} - \frac{1}{2}A^\dagger A\rho_{\rm sf}\,, \tag{10}$$

$$\dot\rho_{\rm ff} = A\rho_{\rm ss}A^\dagger\,, \tag{11}$$

we notice that the master equation describes the original effective Schrödinger equation (5) but with properly normalized states [14]. By construction the time evolution of $\rho_{\rm ss}$ is independent of $\rho_{\rm sf},\rho_{fs}$ and $\rho_{\rm ff}$. Further $\rho_{\rm sf}$ and ρ_{fs} completely decouple from $\rho_{\rm ss}$ and thus can without loss of generality be chosen to be zero; they are not physical and can never be measured. With the initial condition $\rho_{\rm ff}(0)=0$ the time evolution is *solely* determined by $\rho_{\rm ss}$ – as expected for a spontaneous decay process – and is formally given by integrating (11).

3 Time evolution of single kaons

Without loss of generality the initial state can be chosen in the mass-eigenstate basis $\{K_{\rm S},K_{\rm L}\}$. The formal solution of (7) ($\Gamma=\frac{1}{2}(\Gamma_{\rm S}+\Gamma_{\rm L})$ and the numbers $\rho_{\rm SS}+\rho_{\rm LL}=1$) is

$$\rho(t)=\begin{pmatrix}{\rm e}^{-\Gamma_{\rm S}t}\rho_{\rm SS} & {\rm e}^{-{\rm i}\Delta mt-\Gamma t}\rho_{\rm SL} & 0 & 0\\ {\rm e}^{{\rm i}\Delta mt-\Gamma t}\rho_{\rm SL}^* & {\rm e}^{-\Gamma_{\rm L}t}\rho_{\rm LL} & 0 & 0\\ 0 & 0 & F_{\rm L}\rho_{\rm LL} & X^*\\ 0 & 0 & X & F_{\rm S}\rho_{\rm SS}\end{pmatrix}\,, \tag{12}$$

with $F_{\rm S/L}=1-{\rm e}^{-\Gamma_{\rm S/L}t}$ and

$$X=\frac{\sqrt{\Gamma_{\rm S}\Gamma_{\rm L}}}{-{\rm i}\Delta m-\Gamma}(1-{\rm e}^{-{\rm i}\Delta mt-\Gamma t})\rho_{\rm SL}\,.$$

Clearly, we have ${\rm Tr}\rho(t)=1$, and the decay is caused by the environment (treating the neutral kaon in the QFT formalism, the decay would be caused by the QCD vacuum). The surviving part of the single kaon evolving in time is represented by the upper 2×2 block matrix $\rho_{\rm ss}$, the lower one by the decaying part $\rho_{\rm ff}$.

Only properties of the surviving components can be measured. E.g. by a piece of matter an incoming beam is forced to react with the matter via the strong interaction (which is strangeness conserving). If a reaction which can only be caused by a $\bar K^0$ is detected, one records a yes event (Y). If no $\bar K^0$ is detected a no event (N) is recorded (including a K^0 or a decay event). Then matter acts in the very same manner as an ordinary polarizator for photons. Note that an experimenter can *actively* choose the initial state (up to experimental realization), the kind of detector (experimentally very limited) and where to place the detector, i.e. how much "decoherence" the system undergoes, whereas the kind of "decoherence" is given by nature. Note that this "decoherence" is fundamentally different from that in other quantum systems which are stable; there the kind of decoherence depends on the environment, for kaons it is intrinsic to the system.

Consequently, an operator P projecting onto the states $\rho_{\rm ss}$ gives the two probabilities, for Y or N, that a certain state is detected at time t:

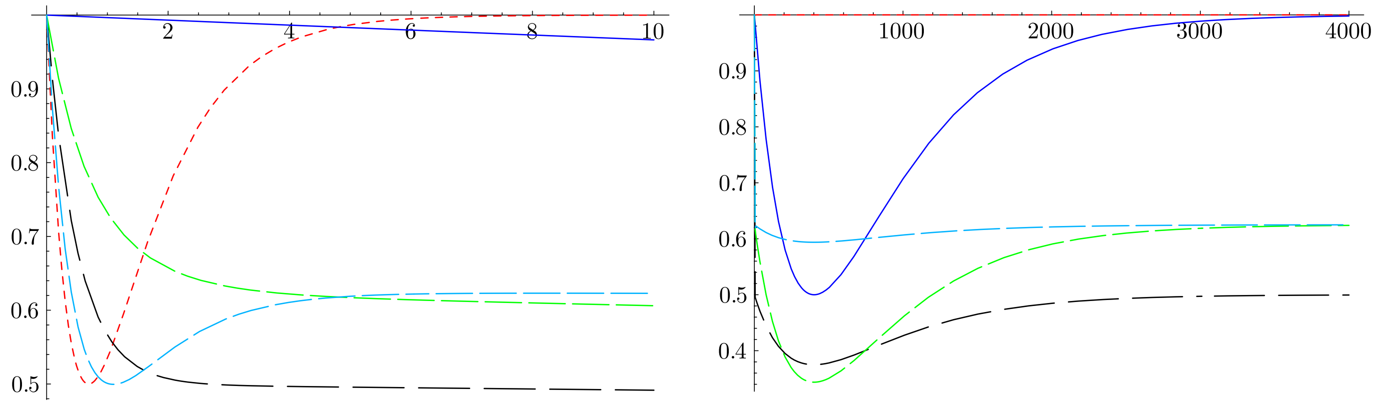
$$\begin{aligned}{\rm Prob}(Y,t)&={\rm Tr}\left(\begin{pmatrix}P & 0\\ 0 & 0\end{pmatrix}\rho(t)\right)={\rm Tr}(P\rho_{\rm ss}(t))\quad\text{and}\\ {\rm Prob}(N,t)&={\rm Tr}\left(\begin{pmatrix}\mathbb{1}-P & 0\\ 0 & \mathbb{1}\end{pmatrix}\rho(t)\right)\\ &={\rm Tr}((\mathbb{1}-P)\rho_{\rm ss}(t))+{\rm Tr}(\rho_{\rm ff}(t))\\ &=1-{\rm Tr}\left(P\varrho_{\rm ss}(t)\right)\,.\end{aligned}$$

Consequently, the expectation value becomes $E_P(t)={\rm Prob}(Y,t)-{\rm Prob}(N,t)=2{\rm Tr}(P\rho_{\rm ss}(t))-1$ and is *solely* determined by the surviving component $\rho_{\rm ss}$!

We considered all possible projectors, and the $\rho_{\rm ff}$ enters in the probabilities only via the trace; thus it is clear that the off diagonal elements of $\rho_{\rm ff}$ are not relevant for any probability we may derive. This leaves a certain ambiguity in defining the decaying components and therefore purity and entanglement. We choose the off diagonal elements of $\rho_{\rm ff}$ in (12) equal to zero because they give the lowest purity values.

Let us now consider the change of the properties of the state $\rho(t)$ with time by considering the purity defined by

$$\begin{aligned}{\rm Tr}\rho(t)^2&={\rm Tr}\left(\rho_{\rm ss}(t)^2\right)+{\rm Tr}\left(\rho_{\rm ff}(t)^2\right)\\ &={\rm Tr}\rho_{\rm ss}(t)^2+({\rm Tr}\rho_{\rm ff}(t))^2\\ &={\rm Tr}\rho_{\rm ss}(t)^2+(1-{\rm Tr}\rho_{\rm ss}(t))^2\\ &=\rho_{\rm SS}^2(1-2{\rm e}^{-\Gamma_{\rm S}t}+2{\rm e}^{-2\Gamma_{\rm S}t})\\ &\quad+\rho_{\rm LL}^2(1-2{\rm e}^{-\Gamma_{\rm L}t}+2{\rm e}^{-2\Gamma_{\rm L}t})+2|\rho_{\rm SL}|^2{\rm e}^{-2\Gamma t}\,.\end{aligned}$$



(a) (b)

Fig. 1. Here the purity $\mathrm{Tr}\,\rho(t)^2$ for single kaons (initially pure) for short (**a**) and longer (**b**) time scales is shown (units in $1/\Gamma_\mathrm{S}$): *dashed* K_S; *solid* K_L; *long dashed* K^0 or $\bar{K}^0$; *green* $1/2|K_\mathrm{S}\rangle + \sqrt{3/4}|K_\mathrm{L}\rangle$; *light blue* $\sqrt{3/4}|K_\mathrm{S}\rangle + 1/2|K_\mathrm{L}\rangle$

Note that the second equality sign is only true if the off diagonal elements of ρ_ff vanish. Otherwise we would add an additional in general time dependent factor to the definition of the purity (for the formal integration of the order 10^{-2}). Again our definition of the purity is only depending on the surviving components. Starting with an arbitrary initial pure state we see that the decay ability of the system leads to a decrease in purity for $t > 0$. For K_S or K_L the purity returns to 1 for $t \to \infty$ depending on the decay constants; see Fig. 1. After a time $t/\tau_{\mathrm{S/L}} = \ln 2$ the minimal purity of 0.5 of a usual qubit system described by a 2×2 density matrix (trace state) is reached. For other superpositions the purity oscillates to a certain final purity which $\neq 1$. For an initial K^0 or $\bar{K}^0$ we reach the minimal purity of 0.375 at time $t/\tau_\mathrm{S} = 401.881$, i.e. about 2/3 of the lifetime of the long-lived state. This is much lower than the purity of a qubit system. Indeed, this decaying system – where only two degrees of freedom can be measured – behaves as regards the purity properties as a system with more degrees of freedom; neutral kaons are more like a double slit system evolving in time [17]. Clearly, we could renormalize the purity by choosing appropriate off diagonal elements of ρ_ff.

The minimal purity which can be reached for this decaying system is 0.333068, obtained by an initially mixed state $(\rho_\mathrm{SS} = 2/3; \rho_\mathrm{SL} = 0; t/\tau_\mathrm{S} = 0.694012)$, and is thus greater than 0.25, the minimal value for a 4×4 density matrix (trace state). Note that in general it depends on the ratio between $\Gamma_\mathrm{S}/\Gamma_\mathrm{L}$ and is therefore intrinsic to the described meson system.

4 The time evolution for two kaons

Any density matrix of a single kaon evolving in time, (12), can be decomposed in the following way:

$$\rho(t) = \sum_{nm} f_{nm}(t)\rho_{nm}|n\rangle\langle m|\,.$$

Clearly, for two kaons in a product state we have

$$\sigma(t) = \sum_{nmlk} f_{nm}(t) f_{lk}(t)\rho_{nm}\rho_{lk}|n\rangle\langle m|\otimes|l\rangle\langle k|\,,$$

and, consequently, any two-kaon state is then given by

$$\sigma(t) = \sum_{nmlk} f_{nm}(t) f_{lk}(t)\sigma_{nmlk}|n\rangle\langle m|\otimes|l\rangle\langle k|\,, \quad (13)$$

where the time dependent weights can be assumed to factorize. In order to do this one has to prove that the projectors commute with the generators of the time evolution under the trace (this was proven in a different formulation in [18]). We can even define a two-particle density matrix depending on the two different times representing the times when the two kaons are measured, i.e.

$$\sigma(t_\mathrm{l}, t_\mathrm{r}) = \mathrm{diag}\{\sigma_\mathrm{ssss}(t_\mathrm{l}, t_\mathrm{r}), \sigma_\mathrm{ssff}(t_\mathrm{l}, t_\mathrm{r}), \sigma_\mathrm{ffss}(t_\mathrm{l}, t_\mathrm{r}), \sigma_\mathrm{ffff}(t_\mathrm{l}, t_\mathrm{r})\}\,,$$

where σ_{iijj} are 4×4 matrices. As a measure of entanglement we want to consider the entanglement of formation which is defined by $\mathcal{E}o\mathcal{F}(\rho) = \min_i \sum_i p_i S(\mathrm{Tr}_\mathrm{l}(|\psi_i\rangle\langle\psi_i|))$, where S is the von Neumann entropy, the trace is taken over one subsystem (left or right), and the ψ_i are the pure state decompositions of ρ. A necessary criterion for entanglement is that the matrix under partial transposition (PT) has at least one negative eigenvalue. Only for bipartite two-level systems PT is also sufficient for detecting all entangled states. For the density matrix under investigation PT acts in the following way:

$$\mathrm{PT}[\sigma(t_\mathrm{l}, t_\mathrm{r})] = \mathrm{diag}\{\mathrm{PT}[\sigma_\mathrm{ssss}(t_\mathrm{l}, t_\mathrm{r})], \mathrm{PT}[\sigma_{ssff}(t_\mathrm{l}, t_\mathrm{r})], \mathrm{PT}[\sigma_{ffss}(t_\mathrm{l}, t_\mathrm{r})], \mathrm{PT}[\sigma_{ffff}(t_\mathrm{l}, t_\mathrm{r})]\}\,.$$

The surviving–surviving block σ_ssss can lead to negative eigenvalues, i.e., can be entangled, while the eigenvalues of the other blocks cannot become negative due to the zero off diagonal elements; the eigenvalues remain unchanged

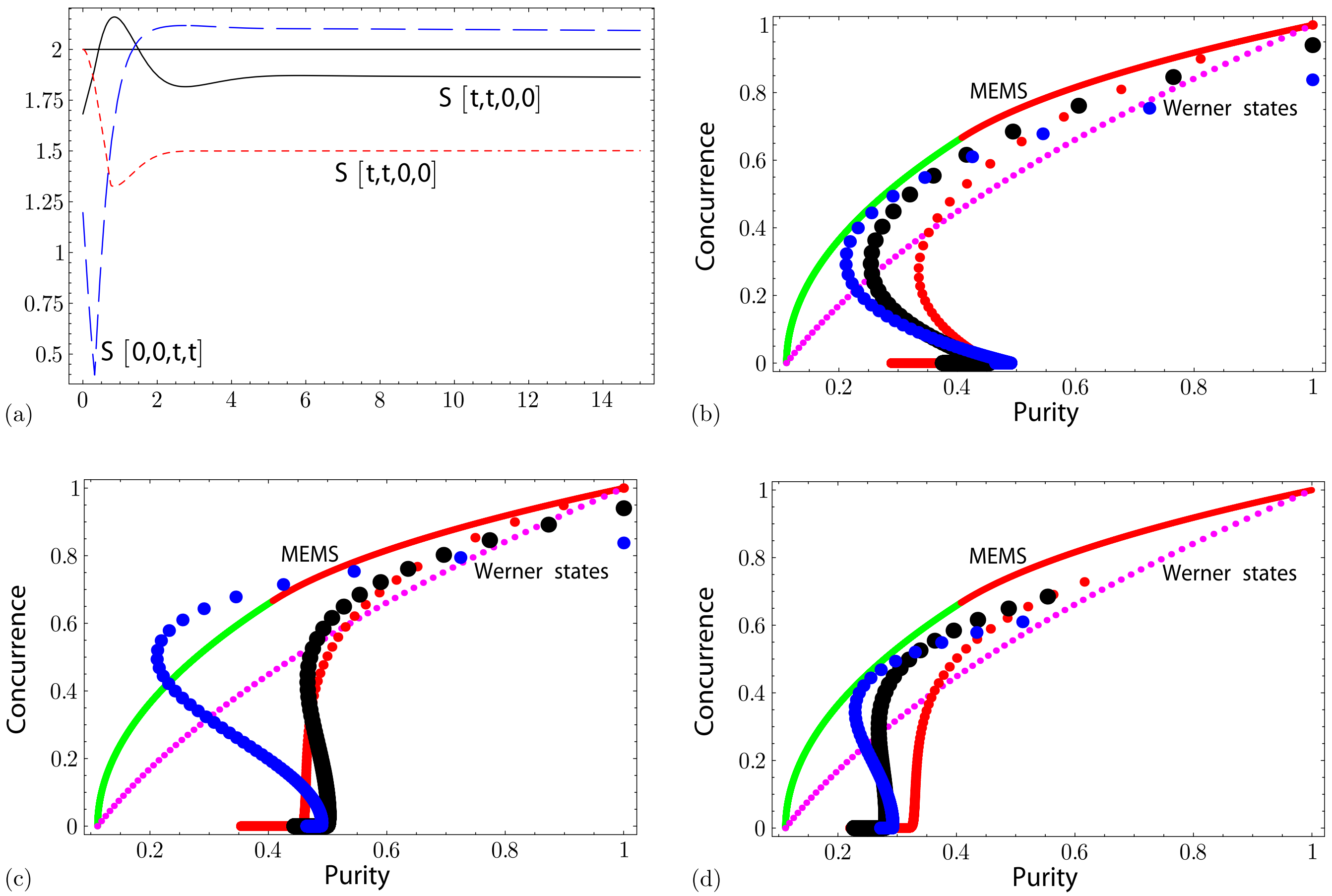


Fig. 2. In **a** is shown the time dependent S-function for ϕ^+ (*dashed*), ξ (*long dashed*) and χ (*solid*) (time in units of $[\Gamma_S/\Delta m]$). For ξ the violation exists up to $1/\Gamma_L$. In **b** and **c** a purity versus concurrence diagram is drawn (purity normalized $(d\mathrm{Tr}\rho^2-1)/(d-1)$ with $d=4$ for bipartite qubits and $d=16$ for bipartite kaons). The *limiting curve* represents the maximally entangled mixed states (MEMS) [22] and the nearly *linear curve* the Werner states for bipartite qubits. The *dots* are drawn for different initial states, and the time proceeds from 0 to 100 with a step width of 0.05 (units as above). The *smallest dots* are for ϕ^+, the *next to smallest dots* for ξ, and the *biggest dots* are for χ. In **b** is shown the change in purity and concurrence for $t_l=t_r=t$, in **c** for $t_l=0$ and $t_r=t$, and in **d** for $t_l=0.3$ and $t_r=t$

under PT. Thus whether the state under investigation is entangled depends only on σ_{ssss}. For 4×4 matrices entanglement of formation is an increasing function of the computable concurrence $\mathcal{C}$, found by Hill and Wootters [19].[2] Thus we can measure entanglement by the concurrence of σ_{ssss}.

To compute concurrence one defines the flipped matrix $\tilde{\sigma}_{\mathrm{ssss}}=(\sigma_y\otimes\sigma_y)\sigma^*_{\mathrm{ssss}}(\sigma_y\otimes\sigma_y)$ where σ_y is the y-Pauli matrix and the complex conjugation is taken in the $K_S K_L$ basis. The concurrence is then given by the formula $\mathcal{C}=\max\{0,\lambda_1-\lambda_2-\lambda_3-\lambda_4\}$, where the λ_i are the square roots of the eigenvalues, in decreasing order, of the matrix $\sigma_{\mathrm{ssss}}\tilde{\sigma}_{\mathrm{ssss}}$.

Let us now consider a general pure state at $t=0$ (with $r_1^2+r_2^2+r_3^2+r_4^2=1$; $\otimes$ omitted):

$$|\psi(0)\rangle = r_1 e^{i\phi_1}|K_S\rangle|K_S\rangle + r_2 e^{i\phi_2}|K_S\rangle|K_L\rangle + r_3 e^{i\phi_3}|K_L\rangle|K_S\rangle + r_4 e^{i\phi_4}|K_L\rangle|K_L\rangle\,. \quad (14)$$

[2] For higher dimensions no computable function of entanglement of formation is known.

Alice and Bob perform their measurements at certain times t_l, t_r, respectively. For a general initial pure state the concurrence is derived to be

$$\mathcal{C}(\sigma_{\mathrm{ssss}}(t_l,t_r)) = 2\left|r_1 r_4 e^{i\phi_1+i\phi_4} - r_2 r_3 e^{i\phi_2+i\phi_3}\right| e^{-\Gamma(t_l+t_r)}\,.$$

It is simply the concurrence of the initial pure state multiplied by the time depending damping factor. For one time equal to zero the decrease in entanglement is lowest.

We choose as projectors $P_{r,l}=|\bar{K}^0\rangle\langle\bar{K}^0|$, and the expectation value becomes after a cumbersome calculation

$$\begin{aligned}
&E_{\bar{K}^0,\bar{K}^0}(t_l,t_r)\\
&\quad = 1 + r_1^2 e^{-\Gamma_S(t_l+t_r)} + r_2^2 e^{-\Gamma_S t_l-\Gamma_L t_r}\\
&\quad + r_3^2 e^{-\Gamma_L t_l-\Gamma_S t_r} + r_4^2 e^{-\Gamma_L(t_l+t_r)}\\
&\quad - r_1^2(e^{-\Gamma_S t_l}+e^{-\Gamma_S t_r}) - r_2^2(e^{-\Gamma_S t_l}+e^{-\Gamma_L t_r})\\
&\quad - r_3^2(e^{-\Gamma_L t_l}+e^{-\Gamma_S t_r}) - r_4^2(e^{-\Gamma_L t_l}+e^{-\Gamma_L t_r})\\
&\quad + 2r_1 r_2(1-e^{-\Gamma_S t_l})\cos(\Delta m t_r+\phi_1-\phi_2)e^{-\Gamma t_r}\\
&\quad + 2r_1 r_3\cos(\Delta m t_l+\phi_1-\phi_3)e^{-\Gamma t_l}(1-e^{-\Gamma_S t_r})
\end{aligned}$$

$$+ 2r_2r_4 \cos(\Delta m t_l + \phi_2 - \phi_4)\mathrm{e}^{-\Gamma t_l}(1 - \mathrm{e}^{-\Gamma_L t_r})$$
$$+ 2r_3r_4(1 - \mathrm{e}^{-\Gamma_L t_l}) \cos(\Delta m t_r + \phi_3 - \phi_4)\mathrm{e}^{-\Gamma t_r}$$
$$+ 2r_1r_4 \cos(\Delta m(t_l + t_r) + \phi_1 - \phi_4)\mathrm{e}^{-\Gamma(t_l+t_r)}$$
$$+ 2r_2r_3 \cos(\Delta m(t_l - t_r) + \phi_2 - \phi_3)\mathrm{e}^{-\Gamma(t_l+t_r)} \,. \quad (15)$$

We notice that for any initial state one always has damping functions from the decay property in this system, different from other two-state systems, and the expectation value converges, for both times to infinity, to +1. For the initial maximally entangled Bell states $\phi^{\pm}$ ($r_2 = r_3 = 0$) the oscillation goes with the sum of the times, different from the maximally entangled Bell states $\psi^{\pm}$ ($r_1 = r_4 = 0$), where the oscillation only depends on the difference of the times. Thus for $\phi^{\pm}$ a violation of the Bell inequality would occur earlier. However, it turns out that for no maximally entangled state a violation can be found by numerically optimizing with different standard methods (none guarantees a global maximum).

For all phases $\phi_i = 0$ we find the value

$$S = 2.1175$$

(state ξ ($r_1 = -0.8335; r_2 = r_3 = -0.2446; r_4 = 0.4308$): $t_1 = t_2 = 0; t_3 = t_4 = 5.77\tau_S$). If we also vary over the phases we obtain a slightly higher value:

$$S = 2.1596$$

(state χ ($r_1 = -0.7823; r_2 = r_3 = 0.1460; r_4 = 0.5877; \phi_1 = -0.2751; \phi_2 = \phi_3 = -0.6784; \phi_4 = 0$): $t_1 = t_2 = 1.79\tau_S; t_3 = t_4 = 0$); see also Fig. 2a. For the above cases the concurrence gives

$$\mathcal{C}(\xi) = 0.84\mathrm{e}^{-\Gamma(t_l+t_r)} \quad \text{and} \quad \mathcal{C}(\chi) = 0.94\mathrm{e}^{-\Gamma(t_l+t_r)} \,.$$

In Fig. 2b–d purity versus concurrence diagrams are drawn. For ϕ^+ we notice that the "decoherence" caused by the decay exceeds the purity-concurrence values of Werner states, which represent an upper limit in this diagram for all possible decoherence modes given by a Lindblad equation for an initially maximally entangled qubit state [20, 21]. An early decay of one kaon, Fig. 2c, exceeds even the purity–concurrence value of maximally entangled mixed bipartite qubit states (MEMS) [22].

To sum up, the initial entanglement decreases with a sum of times, and it goes first hand in hand with a decrease in purity which can then for latter times increase again. For non-maximal entangled state the decrease of purity is much faster than for the maximally entangled states. This seem to help to violate the Bell–CHSH inequality, though the ratio of oscillation to decay is low.

All other meson systems have the same decay rate for both mass-eigenstates, but no *active* measurements are possible due to their fast decay, a necessary condition for any test of local realistic theories versus QM. For B-mesons, the symmetric Bell state ψ^+ violates formally the Bell inequality while ψ^- does not, though both states have the same purity–concurrence behavior. The violation of a Bell inequality depends strongly on the parameters describing these systems, rather than on the amount of entanglement.

5 Conclusions

We show how to treat a single and bipartite decaying neutral kaon system in quantum mechanics and analyze the properties of the corresponding states via purity, entanglement and nonlocality. Only two degrees of freedom at a certain time can be measured, reducing the set of observables and leaving some elements of the state undefined.

Different from photons, nonlocality is for the neutral kaon system a quite "dynamical" concept as correlations of states evolving up to different times are involved. For entangled photons there is no difference in principle whether the correlations are measured after one or several meters. With each measurement the experimenter chooses among two observables: the quasi-spin and the detection time. Consequently, considering Bell inequalities for mesons, (2), one can vary in the quasi-spin space or vary the detection times, or both. If varying in the quasi-spin space and for simplicity choosing all times equal to zero, it has been shown in [5] that there is a connection between nonlocality and the violation of a symmetry in high energy physics, i.e. the $\mathcal{CP}$ symmetry ($\mathcal{C}$ = charge conjugation, $\mathcal{P}$ = parity).

In this work we have discussed the choice of measuring on both sides an anti-kaon versus no anti-kaon at a certain time, which can experimentally be realized via inserting a piece of matter at a certain position from the source (corresponding to the detection time). We find a novel violation of the Bell–CHSH inequality for certain initial states, which are more "robust against decoherence" caused by the decay mechanism. For these states – currently not available by experiments – the concurrence is not maximal, in partial agreement with [23, 24], optimal Bell tests not requiring maximally entangled states for systems with more than two degrees of freedom. A higher amount of entanglement does not necessarily imply an increase of the violation of the Bell inequality under investigation; in fact, the Bell inequality need not be violated at all.

Therefore, these results suggest that for the neutral kaon system nonlocality and entanglement are indeed some distinct quantum features which manifest themselves in a way different than that for bipartite qubit or qutrit systems, and their relation is subtler than one naively expects.

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Note added in proof. As recently found by V. Scarani and myself, ϕ^- violates the Bell inequality slightly; this will be published in a common work [25].

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